Available online at www.sciencedirect.com

ScienceDirect

Journal of Computational and Applied Mathematics 215 (2008) 357–367

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Least-squares spectral element method for non-linear hyperbolic differential equations

B. De Maerschalck[a,*], M.I. Gerritsma[b]

[a] *Von Karman Institute for Fluid Dynamics, Waterloosesteenweg 72, 1640 Sint-Genesius-Rode, Belgium*
[b] *Department of Aerospace Engineering, Delft University of Technology, Kluyverweg 1, 2629 HS Delft, The Netherlands*

Received 23 August 2005

Abstract

The least-squares spectral element method has been applied to the one-dimensional inviscid Burgers equation which allows for discontinuous solutions. In order to achieve high order accuracy both in space and in time a space–time formulation has been applied. The Burgers equation has been discretized in three different ways: a non-conservative formulation, a conservative system with two variables and two equations: one first order linear PDE and one linearized algebraic equation, and finally a variant on this conservative formulation applied to a direct minimization with a QR-decomposition at elemental level. For all three formulations an *h/p*-convergence study has been performed and the results are discussed in this paper.

MSC: 41A10; 65M12; 65M70; 65P40

Keywords: Least-squares spectral elements; Hyperbolic equations; h/p-Convergence

1. Introduction

The least-squares finite element method (LS-FEM) as described in [11–13,1,2] forms an interesting alternative to Galerkin and Petrov–Galerkin weak formulations. In contrast to the Galerkin formulation, LS-FEM does not suffer from the compatibility requirements in mixed/constrained formulations, which implies that no *inf–sup* condition between the approximating function spaces needs to be imposed. Furthermore, least-squares methods transform a well-posed system of partial differential equations into a symmetric positive definite system of algebraic equations, for which efficient iterative solvers like the preconditioned conjugate gradient method (PCG) and multigrid methods exist. The symmetric positive-definiteness is independent of the type of the underlying partial differential equation, which allows for a unified approach of a variety of problems in the engineering and scientific field.

In order to obtain high order accuracy the least-squares formulation has been extended to spectral elements as described in [3,14]. The least-squares spectral element method (LS-SEM) was first presented in [17–19]. Independently the method was developed and investigated in [15,16]. The convergence rate with h-refinement and p-enrichment is comparable to Galerkin methods and so is the accuracy in terms of the L^2- and the H^1-norm. For sufficiently smooth exact solutions the convergence rate with polynomial enrichment is exponential.

* Corresponding author.
E-mail address: demaersc@vki.ac.be (B. De Maerschalck).

0377-0427/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2006.03.060

A space–time formulation allows high order accuracy both in space and in time. In this formulation time is considered as an additional spatial dimension. A space–time LS-FEM has been studied, both numerically and theoretically [8,9]. Independently De Maerschalck and Gerritsma applied the space–time formulation to the LS-SEM [4,7,5]. One can solve the whole space–time domain at once, or for long duration computations one can consider to construct a semi-discrete formulation by solving the solution per time-step on space–time slabs. This semi-discrete formulation is the applied strategy in this paper.

In general, spectral element methods perform best when the underlying exact solution is smooth. The use of spectral methods for hyperbolic systems which allow for discontinuities traditionally has been viewed as problematic and therefore very little work has been done in this field. In [4,7,5] it has been demonstrated that LS-SEM is capable to predict discontinuities at the right place, at the right time and with the correct strength. However, some concerns about the convergence of non-linear systems with discontinuous solutions occurred. It was also observed that in case of convergence, different linearization methods could lead to different solutions. In [6] the role of the Gauss-integration on the convergence has been discussed. In classical spectral element approach it is common to use the same Gauss–Lobatto grid for approximation of the integrals involved as is used for the representation of the solution. In [6] it has been demonstrated that when a sufficiently high GL-grid is used for the integration, discrepancies between different linearization methods can be considerably reduced. However, even with this over-integration, it was still hard to find a fully converged solution for the conservative formulation of the one-dimensional Burgers equation. In this paper some new conservative formulations in the least-squares sense are presented and compared to the previous non-conservative formulation.

2. Least-squares spectral element formulation

2.1. Least-squares formulation

The least-squares formulation is based on the minimization of a *norm-equivalent* functional. Consider the system of first order linear partial differential equations:

$$\mathscr{L}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.1}$$

$$\mathscr{R}\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma \subset \partial\Omega, \tag{2.2}$$

with $\mathscr{L}$ a linear first order partial differential operator and $\mathscr{R}$ the trace operator. It is assumed that the system is well-posed and the operator $(\mathscr{L}, \mathscr{R})$ is a continuous mapping from an underlying function space X onto the space $Y(\Omega) \times Y(\Gamma)$. Without loss of generality, we can set $\boldsymbol{g} = \boldsymbol{0}$ which will be the case in the remainder.

The least-squares formulation seeks to minimize the residual of (2.1) and (2.2) in a certain norm. The norm-equivalent functional becomes

$$\mathscr{J}(\boldsymbol{u}) = \tfrac{1}{2}(\|\mathscr{L}\boldsymbol{u} - \boldsymbol{f}\|^2_{Y(\Omega)} + \|\mathscr{B}\boldsymbol{u} - \boldsymbol{g}\|^2_{Y(\Gamma)}). \tag{2.3}$$

Minimizing $\mathscr{J}$ means: find $\boldsymbol{u} \in X$ such that

$$\lim_{\varepsilon \to 0} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathscr{J}(\boldsymbol{u} + \varepsilon \boldsymbol{v}) = 0 \qquad \forall \boldsymbol{v} \in X. \tag{2.4}$$

Consequently, one can write the necessary condition for a minimizer as: find $\boldsymbol{u} \in X$ such that

$$\mathscr{B}(\boldsymbol{u}, \boldsymbol{v}) = \mathscr{F}(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in X, \tag{2.5}$$

where $\mathscr{B}$: $X \times X \to \mathbb{R}$ is a symmetric, continuous bilinear form, defined by $\mathscr{B}(\boldsymbol{u}, \boldsymbol{v}) = (\mathscr{L}\boldsymbol{u}, \mathscr{L}\boldsymbol{v})_{Y(\Omega)} + (\mathscr{R}\boldsymbol{u}, \mathscr{R}\boldsymbol{v})_{Y(\Gamma)}$ and $\mathscr{F} \in X'$: $X \to \mathbb{R}$ is a continuous linear functional, defined by $\mathscr{F}(\boldsymbol{v}) = (\boldsymbol{f}, \mathscr{L}\boldsymbol{v})_{Y(\Omega)} + (\boldsymbol{g}, \mathscr{R}\boldsymbol{v})_{Y(\Gamma)}$.

The inclusion of the boundary residual in (2.3) allows for the use of minimization spaces $X(\Omega)$ that are not constrained to satisfy the boundary condition (2.2), i.e., the boundary conditions are enforced weakly through the variational principle. This is advantageous whenever the boundary condition is difficult to satisfy computationally. In the present work the functions belonging to the space $X(\Omega)$ are required to satisfy the boundary conditions. Consequently, the boundary terms in (2.3) can be omitted and the boundary conditions are enforced strongly on candidate minimizers.

The H^1- and L^2-spaces are particularly suitable as function spaces X and Y, respectively, in least-squares finite or spectral element methods with strongly imposed boundary conditions. Assume that the following relation holds:

$$C_1\|\boldsymbol{u}\|_{H^1(\Omega)} \leqslant \|\mathscr{L}\boldsymbol{u}\|_{L^2(\Omega)} \leqslant C_2\,\|\boldsymbol{u}\|_{H^1(\Omega)} \quad \forall \boldsymbol{u} \in X = \{\boldsymbol{u} \in H^1(\Omega) \mid \mathscr{R}\boldsymbol{u} = 0 \text{ on } \Gamma\}, \tag{2.6}$$

where X represents the space of functions which already satisfy the homogeneous boundary condition and for which the function itself and its first derivatives is square integrable over the domain Ω. If the differential operator $\mathscr{L}$ is linear one can easily show that the following a priory error estimate holds:

$$C_1\|\boldsymbol{u} - \boldsymbol{u}_{\text{exact}}\|_{H^1(\Omega)} \leqslant \|\mathscr{L}\boldsymbol{u} - \boldsymbol{f}\|_{L^2(\Omega)} \leqslant C_2\,\|\boldsymbol{u} - \boldsymbol{u}_{\text{exact}}\|_{H^1(\Omega)}. \tag{2.7}$$

However, for non-linear differential operators no such estimate is available.

In LS-SEM the variational problem is restricted to a finite-dimensional subspace $X^h \subset X$ parameterized by h, h may refer to a characteristic mesh width, a polynomial degree or a combination of both. The discrete variational problem is then given by: Seek $\boldsymbol{u}^h \in X^h$, such that

$$\mathscr{B}(\boldsymbol{u}^h, \boldsymbol{v}^h) = \mathscr{F}(\boldsymbol{v}^h) \quad \forall \boldsymbol{v}^h \in X^h. \tag{2.8}$$

2.2. Spectral elements

In order to obtain a discrete variational problem the domain is decomposed into a finite number of non-overlapping quadrilateral elements Ω_e with $\Omega = \bigcup_{e=1}^{N_c} \Omega_e$, with N_c the number of elements.

In spectral element methods each element is mapped onto the *parent* element via an isoparametric mapping, i.e., a unit line $[-1, 1]$ in one dimension and the bi-unit square, $[-1, 1] \times [-1, 1]$ in two dimensions. Within the parent element the approximate solution is expanded with respect to continuous basis functions

$$u_e^N(\xi) = \sum_{i=0}^{N} u_i^h\, h_i(\xi), \tag{2.9}$$

with $\xi = \chi_e^{-1}(x)$ the local coordinate of x in the parent element, $-1 \leqslant \xi \leqslant 1$, and u_i^h the $N+1$ unknown coefficients in the expansion. The basis functions consist of Lagrangian interpolants through the Gauss–Lobatto–Legendre (GLL) collocation points

$$h_i(\xi) = \frac{(\xi^2 - 1)\mathrm{d}L_N(\xi)/\mathrm{d}\xi}{N(N+1)L_N(\xi)(\xi - \xi_i)}. \tag{2.10}$$

The $N+1$ GLL-points, ξ_i, are the roots of the first derivative of the Legendre polynomial of degree N, extended with the boundary nodes of the parent element. In [5] Chebyshev polynomials have been used instead of Legendre polynomials.

In multiple dimensions and in a space–time formulation, the basis functions can be obtained in terms of tensor products

$$u^h(\xi, \eta) = u^{M,N}(\xi, \eta) = \sum_{i=0}^{M}\sum_{j=0}^{N} u_{i,j}^h\, h_i(\xi)\, h_j(\eta), \tag{2.11}$$

with M the polynomial degree in the ξ-direction and N the order in the η-direction.

The integrals in (2.8) are evaluated numerically using Gauss–Lobatto integration:

$$\int_{-1}^{1} f(\xi)\ \mathrm{d}\xi \approx \sum_{i=0}^{P} w_i\ f(\xi_i), \tag{2.12}$$

with ξ_i the $P+1$ GLL-roots and w_i the GLL-weights

$$w_i = \int_{-1}^{1} h_i(\xi)\,\mathrm{d}\xi. \tag{2.13}$$

Generally, the same GLL-roots are used for the evaluation of the integrals as for the representation of the solution, i.e., $P = N$. However, in [6] the role of P on the convergence has been discussed. For non-linear systems of equations with possible discontinuities in the exact solution it is advised to do the integration on a finer mesh. Often it is sufficient to take only a few GLL-points more for the integration, see [6] for details.

3. Non-linear hyperbolic equations

In the next sections we will compare different weak formulations for the one-dimensional nonlinear Burgers equation of the form

$$\mathscr{L}u = 0 \quad \text{on } \Omega_{\text{st}} = \{(x,t) \mid 0 \leqslant x \leqslant L,\ 0 \leqslant t \leqslant T\}, \tag{3.1}$$

with

$$\mathscr{L}u = \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x}, \tag{3.2}$$

and initial condition $u(x,0) = u_{\text{in}}(x)$ and a Dirichlet boundary condition at $x = 0$.

3.1. Non-conservative formulation

Before applying the discrete variational problem (2.8)–(3.1), the differential operator $\mathscr{L}u$ is linearized. Call u_0 the solution at the previous iteration step and $\delta u = u - u_0$ the difference between two consecutive steps, then one can linearize $\mathscr{L}u$ around u_0

$$\mathscr{L}u \approx \mathscr{L}u_0 + \delta u(\nabla_u \mathscr{L}u)_{u_0} + \mathscr{O}(\delta u^2) \approx \mathscr{L}u_0 + \mathscr{L}'\delta u. \tag{3.3}$$

$\mathscr{L}'\delta u$ for (3.2) is then given by

$$\mathscr{L}'\delta u = \delta u(\nabla_u \mathscr{L}u)_{u_0} = \frac{\partial \delta u}{\partial t} + 2u_0\frac{\partial \delta u}{\partial x} + 2\frac{\partial u_0}{\partial x}\delta u. \tag{3.4}$$

This linearization is known as Newton's method. An alternative linearization is obtained by Picard linearization:

$$\mathscr{L}u \approx \frac{\partial u}{\partial t} + 2u_0\frac{\partial u}{\partial x}. \tag{3.5}$$

In the presented work Newton's method has been applied. The least-squares functional for the linearized Burgers equation is then given by

$$\mathscr{J}(\delta u) = \tfrac{1}{2}\|\mathscr{L}'\delta u + \mathscr{L}u_0\|_{L^2(\Omega)}. \tag{3.6}$$

This formulation works well for smooth underlying exact solutions. If one works out the Gauss-type integrations carefully, also for non-smooth solutions a fully converged solution can be obtained [6].

An *h*/*p*-convergence study has been performed for two different test cases. The first test case is the advection of a small single cosine hill, where the calculation is stopped before a shock develops. The initial condition is given by $u_{\text{in}}(x) = 1 - 0.01\cos(2\pi x)$, for $0 \leqslant x \leqslant 1$ and $u_{\text{in}}(x) = 0.99$ in the remainder of the domain. The calculation is stopped ad $T = 1$. For this test case the exact solution is smooth in the whole space–time domain. The second test case is a discontinuous step function with initial condition a step at $x = 0.5$. Left from the discontinuity the solution is constant and equal to one, right the solution is equal to 0.5. Also for this test case the final time level is set to $T = 1$. For both the test cases the length of the spatial domain is 4.

Figs. 1 and 2 show the convergence of the error and the residual in case of *h*-refinement and *p*-enrichment. The error and the residual are both measured in the L^2-norm

$$\|\varepsilon\|_0 = \|\varepsilon\|_{L^2} = \left(\int_0^L (u^h(x,T) - u_{\text{exact}}(x,T))^2\,\mathrm{d}\Omega\right)^{1/2}, \tag{3.7}$$

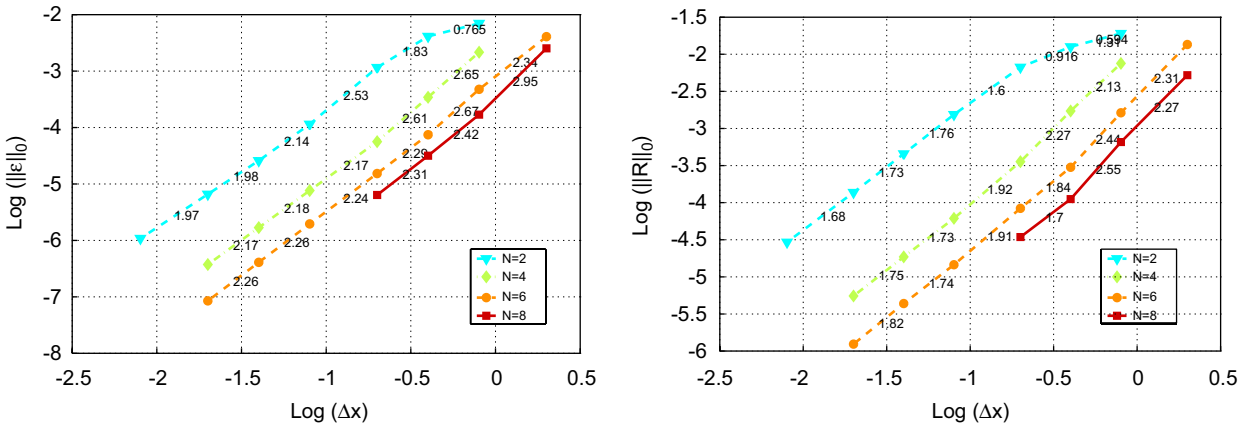

Fig. 1. h-Convergence of the error and residual in the L^2-norm for the smooth test case with the non-conservative formulation.

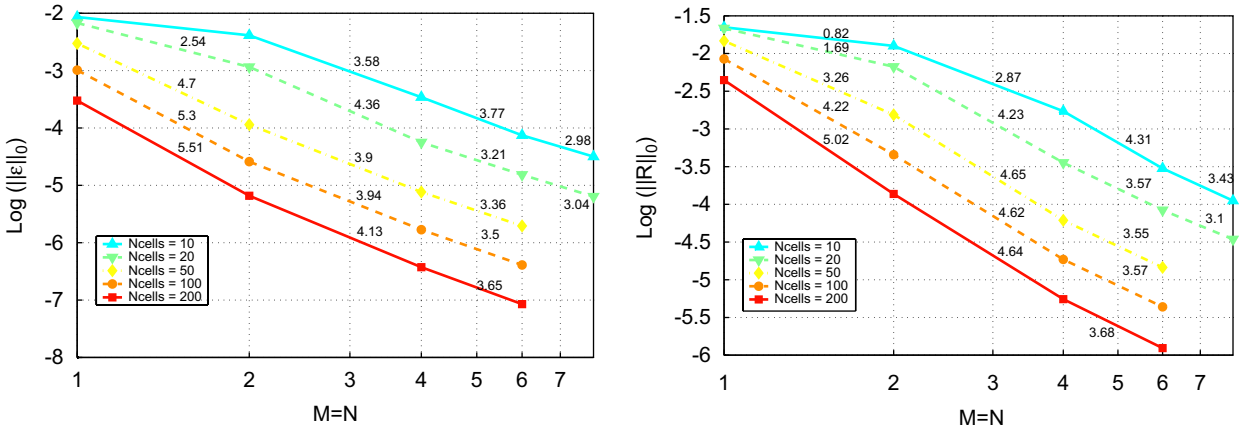

Fig. 2. p-Convergence of the error and residual in the L^2-norm for the smooth test case with the non-conservative formulation.

with $u^h(x, T)$ the LS-SEM solution at the final time level and $u_{\text{exact}}(x, T)$ the exact solution, and for the residual

$$\|R\|_0 = \|R\|_{L^2} = \left(\int_{\Omega_{\text{st}}} (\mathscr{L} u^h)^2 \, \mathrm{d}\Omega \right)^{1/2}, \tag{3.8}$$

with Ω_{st} the entire space–time domain. The numbers in the graphs present the local convergence rates. Since the solution is smooth, but not infinitely smooth the convergence is in all cases algebraic.

Figs. 3 and 4 show the same convergence study, but now for the second test case with discontinuous solutions. It is obvious that the residual in the L^2-norm does no longer converge. Instead the right graphs in Figs. 3 and 4 show a diverging behavior. Remember from (2.7) that the L^2-norm of the residual gives an estimate on the error in the H^1-norm. Although (2.7) is only applicable to linear differential equations, we cannot guarantee with the current approach that we even solve the linearized Burgers' equation. For this test case we see convergence of the error in the L^2-norm, so not matter the behavior of the residual, we clearly approximate the exact solution in the L^2-norm. However, in general we do not know the exact solution and we therefore have no way of establishing convergence of the error in the L^2-norm. If we apply (2.7) to the linearized equations we can ensure that convergence in the L^2-norm is guaranteed if convergence of the residual norm is observed. Only this norm of the residual is computable and therefore we need to find a formulation in which both the error and residual converge with h- or p-refinement.

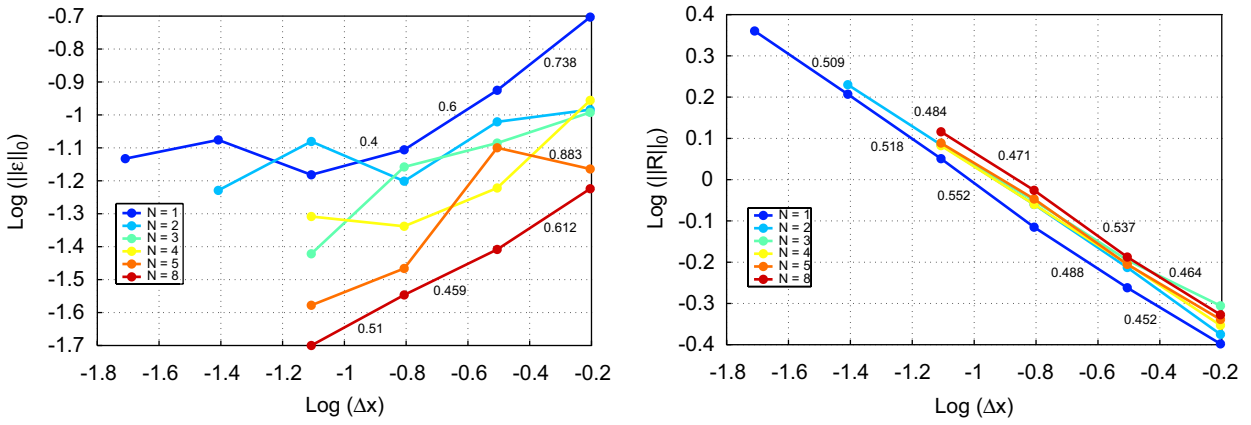

Fig. 3. h-Convergence of the error and residual in the L^2-norm for the discontinuous test case with the non-conservative formulation.

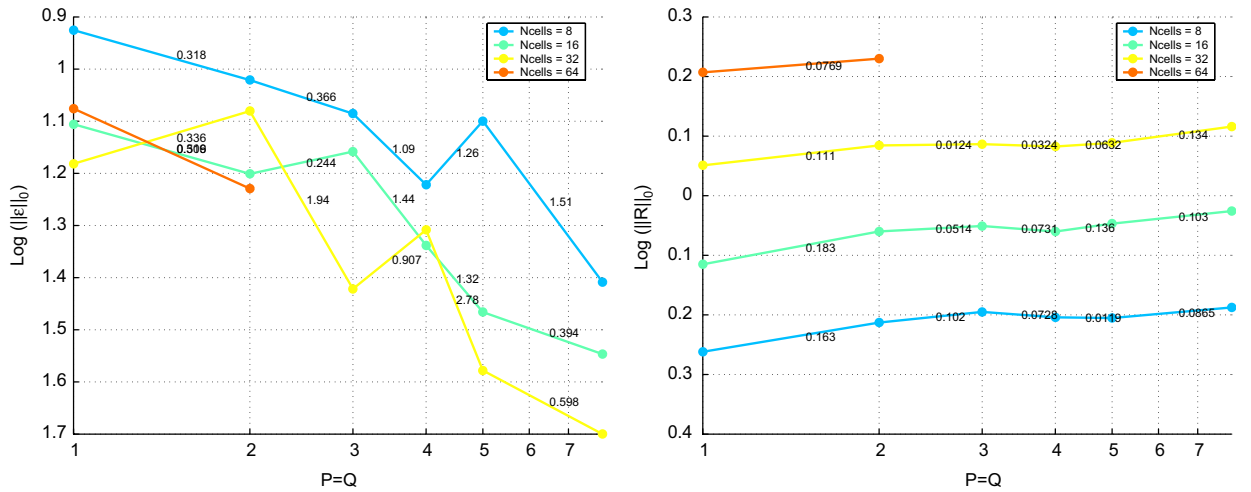

Fig. 4. p-Convergence of the error and residual in the L^2-norm for the discontinuous test case with the non-conservative formulation.

3.2. Conservative formulation

In many numerical schemes for conservation laws, a conservative formulation is desired. It is possible to rewrite the Burgers equation into a system of equations by adding the flux as additional unknown

$$\mathscr{L}(u, F) = \boldsymbol{f} \Longleftrightarrow \begin{cases} u_t + F_x = 0, \\ F = F(u), \end{cases} \tag{3.9}$$

with $u_t = \partial u/\partial t$, $F_x = \partial F/\partial x$ and $F(u) = u^2$. The first equation is a linear first order partial differential equation and the second equation is a non-linear algebraic equation for the flux. One can linearize this second equation around u_0 by

$$F(u) \approx F(u_0) + J_0\,(u - u_0) = J_0\,u + (F(u_0) - J_0\,u_0), \tag{3.10}$$

with, for a system of equations, J_0 the Jacobian matrix of $F(u)$ evaluated in $u = u_0$, thus in this case $J_0 = 2u_0$ and $(F(u_0) - J_0\,u_0) = -u_0^2$.

This formulation for the Burgers equation has been discussed before [4,6] however, it was hard to find a fully converged solution, even when over-integration for the Gauss-integrations was applied. However, for these calculations both the solution u and the flux F were approximated by equal order polynomials. Reconsider for a moment the

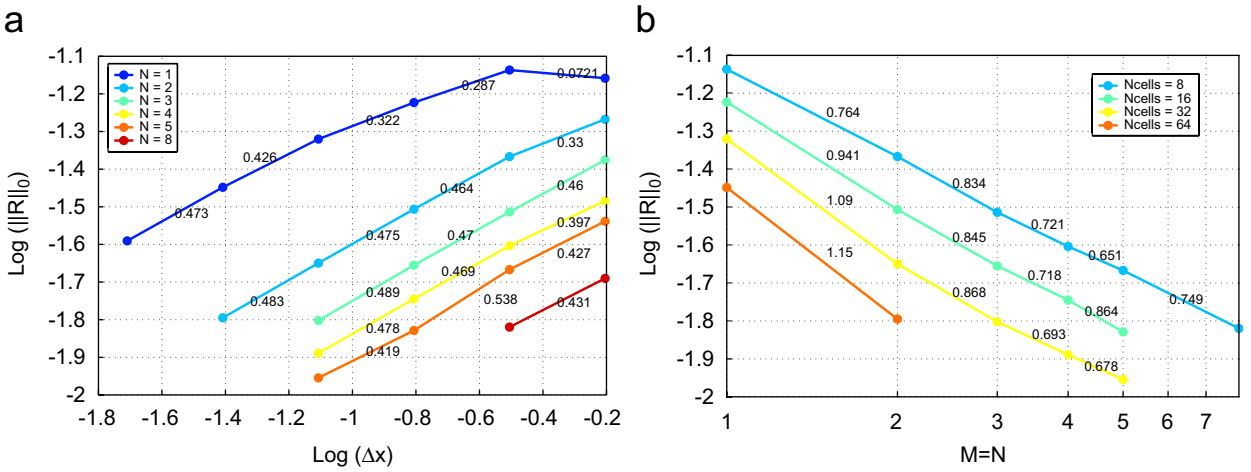

Fig. 5. h- and p-convergence of the residual in the L^2-norm for the discontinuous test case with the conservative formulation using a higher order flux approximation. (a) h-refinement; (b) p-enrichment.

(linearized) algebraic equation

$$F - 2u_0\, u = -u_0^2. \tag{3.11}$$

Suppose that u is approximated by piecewise polynomials of order N_u. Then also u_0 is a piecewise polynomial of the degree N_u and both the product of u and u_0, and the right-hand side of (3.11) are piecewise polynomials of degree $2N_u$. By choosing now the polynomial degree for the approximation of the flux equal to N_u, one might loose accuracy and the method may become unstable. From (3.11) one can notice that it is necessary to choose $N_F = 2\,N_u$ to fully represent the fluxes.

Fig. 5 shows the convergence of the residual for the discontinuous test case, when the discrete variational principle is applied to the linearized formulation (3.9). Both for the h-refinement as for the p-type enrichment the L^2-norm of the residual converges algebraically. For the error one could find analogous behavior as for the non-conservative formulation. Keep in mind that for this definition of the L^2-norm of the residual holds:

$$\|R\|_{L^2} = \left(\int_{\Omega_{\mathrm{st}}} (\mathscr{L}(u,F) - f)^2\,\mathrm{d}\Omega\right)^{1/2} = \left(\int_{\Omega_{\mathrm{st}}} (u_t + F_x)^2\,\mathrm{d}\Omega + \int_{\Omega_{\mathrm{st}}} (F - u^2)^2\,\mathrm{d}\Omega\right)^{1/2} \tag{3.12}$$

This conservative formulation has also the advantage that is very straightforward to make the formulation discontinuous. For the convergence results of the conservative formulation using a higher order flux approximation, the solutions u was made discontinuous in spatial direction while the flux was made discontinuous in temporal direction.

3.3. Conservative formulation with QR-decomposition at element level

In the conservative formulation the flux has been introduced as an extra unknown. Introducing this new variable more than doubles the degrees of freedom. If we would apply this formulation to a multi-dimensional problem the system becomes too large compared to other types of solvers and the element matrices would grow dramatically. For the two-dimensional Euler equations this would already require 12 unknowns and equations instead of four equations and unknowns used in conventional Euler solvers. In addition, the fluxes that need to be introduced require much more refined grids as argued above. It would therefore be interesting if one could reduce the size of the element matrices before assembly in the global matrix.

The linearized form of equation (3.9) can be written as

$$\mathscr{L}(F,u) = \begin{bmatrix} I & -J_0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial t} \end{bmatrix} \begin{pmatrix} F \\ u \end{pmatrix} = \begin{pmatrix} F_F \\ F_u \end{pmatrix}, \tag{3.13}$$

with $F_F = -u_0^2$ and $F_u = 0$. This operator can be discretized directly, [10], which leads to

$$\mathcal{L}(\boldsymbol{F}^h, \boldsymbol{u}^h) = \begin{bmatrix} I^h & -J_0^h \\ D_x^h & D_t^h \end{bmatrix} \begin{pmatrix} \boldsymbol{F}^h \\ \boldsymbol{u}^h \end{pmatrix} = \begin{pmatrix} F_F^h \\ F_u^h \end{pmatrix}, \tag{3.14}$$

where u^h and F^h are discretized by $\boldsymbol{u}^h = \sum_{i=1}^{N_u} \boldsymbol{u}_i^h \, h_i^u(\boldsymbol{\xi})$ and $\boldsymbol{F}^h = \sum_{i=1}^{N_F} \boldsymbol{F}_i^h \, h_i^F(\boldsymbol{\xi})$. Then $[D_x^h]_{ij} = \partial h_j^F / \partial x|_{\xi_i}$ and $[D_t^h]_{ij} = \partial h_j^u / \partial t|_{\xi_i}$ with $\boldsymbol{\xi_i}$ the N_u GLL-roots, and $[J_0^h]_{ij} = 2u_0(\boldsymbol{\xi_i}) h_j^u(\boldsymbol{\xi_i})$ with in this case $\boldsymbol{\xi}_i$ the N_F GLL-roots. I^h is a unit matrix with dimension $N_F \times N_F$.

Next a QR-decomposition is performed at elemental level for the left part of the discrete element matrix in (3.14)

$$\begin{pmatrix} I^h \\ D_x^h \end{pmatrix} = Q \begin{pmatrix} R \\ 0 \end{pmatrix}. \tag{3.15}$$

The matrix R is an upper triangular matrix with positive entries on the diagonal. Premultiplying (3.14) with Q^{T} results in

$$\begin{bmatrix} R & A \\ 0 & B \end{bmatrix} \begin{pmatrix} \boldsymbol{F}^h \\ \boldsymbol{u}^h \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}, \tag{3.16}$$

with

$$\begin{pmatrix} A \\ B \end{pmatrix} = Q^{\mathrm{T}} \begin{pmatrix} -J_0^h \\ D_t^h \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c \\ d \end{pmatrix} = Q^{\mathrm{T}} \begin{pmatrix} F_F^h \\ F_u^h \end{pmatrix}. \tag{3.17}$$

The problem is now reduced to solve

$$B\boldsymbol{u}^h = d \tag{3.18}$$

in the least-squares sense. Notice that the QR-decomposition is independent of the solution of the previous iteration step, u_0. Therefore, when solving the solution iteratively, it is sufficient to calculate Q^{T} and R only once. Once u^h is solved for all the element, the fluxes can be obtained by

$$\boldsymbol{F}^h = R^{-1}(-Au^h + c). \tag{3.19}$$

Since R is an upper-triangular matrix, obtaining R^{-1} is straightforward.

Fig. 6 shows the results for this formulation applied to the discontinuous test case. The convergence rates for the h-type refinements are comparable to the previous conservative formulation.

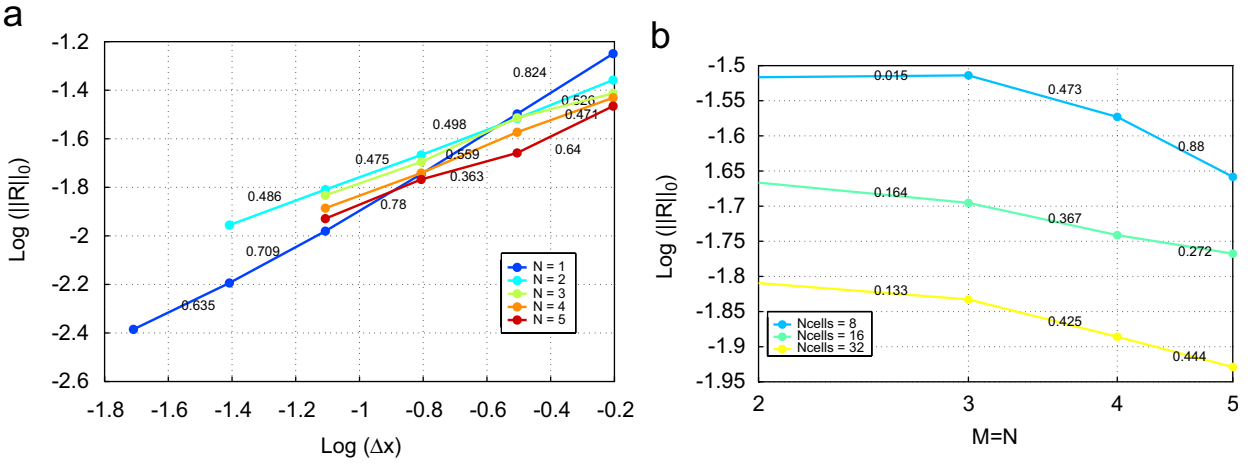

Fig. 6. h- and p-convergence of the residual in the L^2-norm for the discontinuous test case with the conservative formulation with QR-decomposition. (a) h-refinement; (b) p-enrichment.

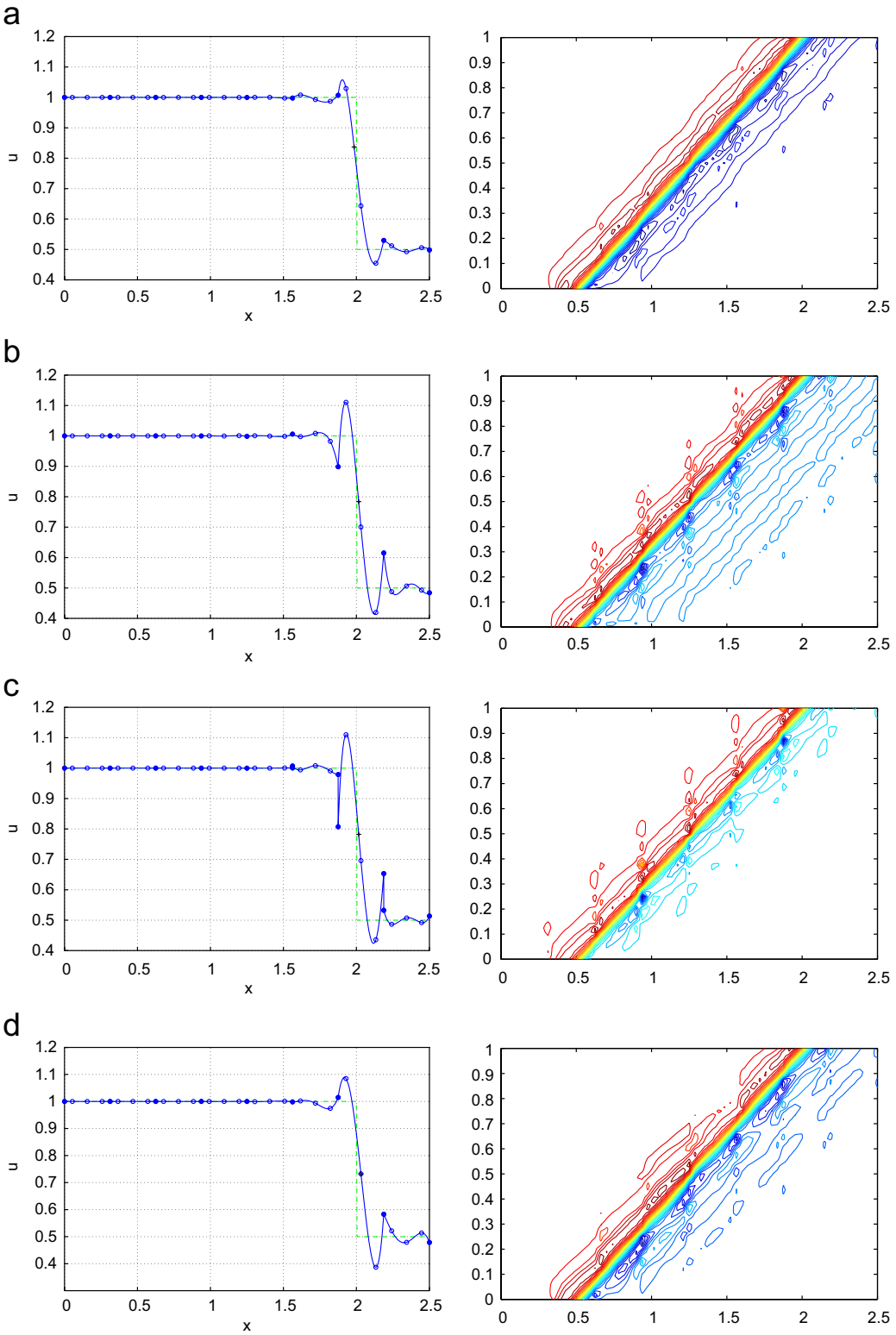

Fig. 7. Final (left) and full space–time (right) LS-SEM solution for the discontinuous test case for the different formulations (number of cells: 8×8, fourth order in space and time). (a) Non-conservative formulation. (b) Continuous conservative formulation. (c) Discontinuous conservative formulation. (d) Conservative formulation with QR-decomposition (continuous).

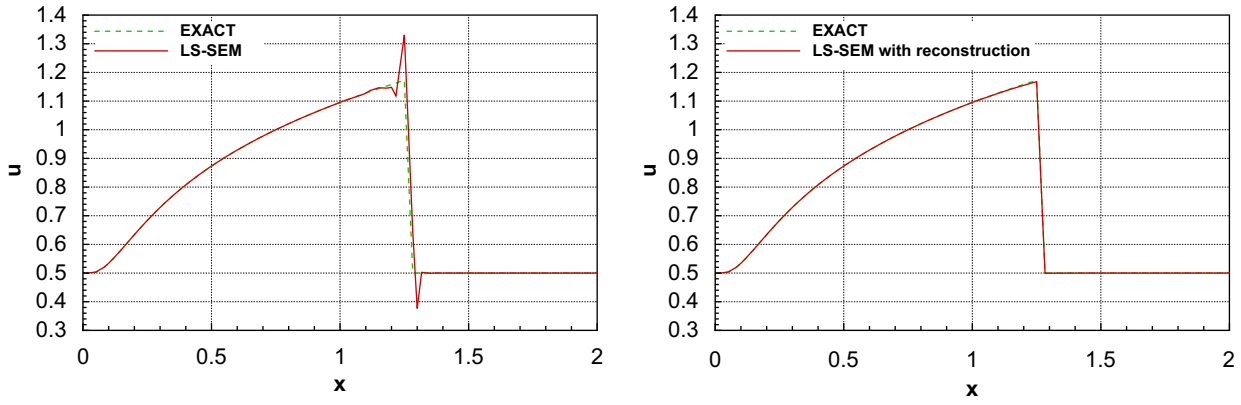

Fig. 8. Original LS-SEM solution (left) and LS-SEM approximation with postprocessing reconstruction (right) applied to the strong non-linear advection equation with initial cosine hill. Number of cells: $N_c = 20$ with polynomial order 4 in space and time.

Fig. 7 shows the final and full space–time solution for all the above discussed formulations. For the conservative formulation a comparison is made between a fully continuous formulation and a discontinuous formulation where $\boldsymbol{u}^h$ is only continuous in the temporal direction and the flux $\boldsymbol{F}^h$ is set continuous only in spatial direction. For all computations the grid consists of 8×8 cells, all fourth order in space and time. Notice the spurious wiggles around the discontinuities. For all formulations these wiggles only occur right in front and behind of the shock. They do not intend to pollute the whole space–time domain. Therefore, these discrepancies can be reduced by a smart adaptive *h/p*-refinement in combination with a postprocessing algorithm as discussed in [4,7,5]. Fig. 8 shows an example of such a postprocessing reconstruction method as explained in detail in these references. For these plots the LS-SEM is applied to a strong non-linear advection equation, $\partial q/\partial t + \partial q^5/\partial x = 0$, with an initial cosine hill. Twenty fourth-order element are used. The left plot shows the original LS-SEM solution, while the right plot shows the result of a simple postprocess algorithm applied to the original numerical solution. Notice that for none of the formulations additional diffusive terms have been added to render a stable scheme. This makes the scheme low diffusive which makes it suitable for, e.g., long time integration.

4. Conclusions

The least-squares spectral element formulation has been applied to the one-dimensional Burgers equation with a smooth and discontinuous underlying exact solution. Three different ways to discretize this conservation law have been proposed. For the non-conservative formulation no additional equations and unknowns have to be introduced. However, *h/p*-convergence displays a divergent behavior for the residual measured in the L^2-norm. Therefore a conservative formulation is proposed. In this formulation the one-dimensional Burgers equation is rewritten as a system consisting of a linear first order partial differential equation and a non-linear algebraic equation for the flux, which has been introduced as an extra variable. For this formulation the residual converges algebraically. This second formulation allows for a straightforward discontinuous approach. One has to take care to approximate the different unknowns by a suitable polynomial degree. Therefore, for a one-dimensional problem the degrees of freedom more than double compared to the non-conservative formulation. For multi-dimensional systems of equations, it gets even worse and the element matrices become very big. In order to keep the size of the global discrete system manageable, a reduction in the degrees of freedom in the global system is obtained by performing a QR-decomposition at elemental level. For this formulation, the convergence rates for the error and the residual both measured in the L^2-norm are comparable to the full conservative formulation. The expensive QR-decomposition only has to be applied once and can be re-used in all non-linear iterations and all time steps. The use of the QR-decomposition to eliminate the fluxes leads to a global discrete system with the size of the non-conservative formulation, but displays the convergence of the residual norm of the conservative formulation.

Notice that for none of the schemes a stabilization technique is required to render a stable, high order scheme. This makes the scheme very low diffusive.

References

[1] P.B. Bochev, Finite element methods based on least-squares and modified variational principles, Technical Report, POSTECH, 2001.
[2] P.B. Bochev, M.D. Gunzburger, Finite element methods of least-squares type, SIAM Rev. 27 (1998) 789–837.
[3] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, Spectral Methods in Fluid Dynamics, Springer Series in Computational Physics, Springer, Berlin, Heidelberg, London, 1987.
[4] B. De Maerschalck, Space–time least-squares spectral element method for unsteady flows—application and evaluation for linear and non-linear hyperbolic scalar equations, Master Thesis, Delft University of Technology, Delft, The Netherlands, ⟨http://www.hsa.lr.tudelft.nl/∼bart⟩, 2003.
[5] B. De Maerschalck, M.I. Gerritsma, The use of the Chebyshev approximation in the space–time least-squares spectral element method, Numer. Algorithms 38 (2005) 173–196.
[6] B. De Maerschalck, M.I. Gerritsma, Higher-order Gauss–Lobatto integration for nonlinear hyperbolic equations, J. Sci. Comput. 27 (1–3) (2006) 201–214.
[7] B. De Maerschalck, M.I. Gerritsma, M.M.J. Proot, Space–time least-squares spectral elements for convection dominated unsteady flows, AIAA J. 44 (2006) 558–565.
[8] H. De Sterck, T.A. Manteuffel, S.F. McCormick, L. Olson, Least-squares finite element methods and algebraic multigrid solvers for linear hyperbolic PDEs, SIAM J. Sci. Comput. 26 (1) (2004) 31–54.
[9] H. De Sterck, T.A. Manteuffel, S.F. McCormick, L. Olson, Numerical conservation properties of $H(div)$-conforming least-squares finite element methods applied to the Burgers equation SIAM J. Sci. Comput. 26 (2005) 1573–1597.
[10] W. Hoitinga, R. de Groot, M.I. Gerritsma, Direct minimization of the least squares spectral element functional—Part I: direct solver, (http://www.lr.tudelft.nl/em/wijnand), 2003.
[11] B.N. Jiang, The Least-Squares Finite Element Method, Theory and Applications in Computational Fluid Dynamics and Electromagnetics, Springer, Berlin, 1998.
[12] B.N. Jiang, On the least-squares method, Comput. Methods Appl. Mech. Eng. 152 (1998) 239–257.
[13] B.N. Jiang, L.A. Povinelli, Least-squares finite element method for fluid dynamics, Comput. Methods Appl. Mech. Eng. 81 (1990) 13–37.
[14] G.E. Karniadakis, S.J. Spencer, Spectral/hp Element Methods for CFD, Oxford University Press, Oxford, 1999.
[15] J.P. Pontaza, J.N. Reddy, Spectral/*hp* least squares finite element formulation for the Navier–Stokes equation, J. Comput. Phys. 190 (2) (2003) 523–549.
[16] J.P. Pontaza, J.N. Reddy, Space–time coupled spectral/*hp* least squares finite element formulation for the incompressible Navier–Stokes equation, J. Comput. Phys. 197 (2) (2004) 418–459.
[17] M.M.J. Proot, The least-squares spectral element method, Ph.D. Thesis, Department of Aerospace Engineering, Delft University of Technology, Delft, The Netherlands, February 2003.
[18] M.M.J. Proot, M.I. Gerritsma, A least-squares spectral element formulation for the Stokes problem, J. Sci. Comput. 17 (2002) 285–296.
[19] M.M.J. Proot, M.I. Gerritsma, Least-squares spectral elements applied to the Stokes problem, J. Comput. Phys. 181 (2002) 454–477.